Math 112: Introductory Real Analysis

§ Continuity

- Limits of functions

Def Let X and Y be metric spaces. Suppose $E \subseteq X$, p is a limit point of E, and $f: E \to Y$ is a function. (q is called the limit of f at p.)

We write $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon \quad \text{for all } x \in E \text{ for which } 0 < d_X(x,p) < \delta.$$

Rmk $p \in X$, but p need not be a point of E.

Even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

The above definition can be reformulated in terms of limits of sequences:

Thm $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n\to\infty} p_n = p$.

proof) ($\Rightarrow$) Suppose $\lim_{x \to p} f(x) = q$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ for all $x \in E$ with $0 < d_X(x,p) < \delta$.

If $\{p_n\}$ is a sequence in E such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$, then there exists N such that $0 < d_X(p_n, p) < \delta$ for all $n \geq N$, and hence $d_Y(f(p_n), q) < \varepsilon$ for all $n \geq N$.

Therefore, $\lim_{n\to\infty} f(p_n) = q$.

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($\Leftarrow$) Suppose $\lim_{x\to p} f(x) \neq q$. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ for which $0 < d_X(x,p) < \delta$ but $d_Y(f(x), q) \geq \varepsilon$.

Taking $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, \cdots)$, we thus find a sequence $\{p_n\}$ in E for which $0 < d_X(p_n, p) < \frac{1}{n}$ (hence $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$) but $d_Y(f(p_n), q) \geq \varepsilon$ (hence $\lim_{n\to\infty} f(p_n) \neq q$). ■

Cor If f has a limit at p, this limit is unique.

Cor Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$.

Then (a) $\lim_{x\to p} (f+g)(x) = A+B$

(b) $\lim_{x\to p} (fg)(x) = AB$

(c) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$

defined only at those points of E at which $g(x) \neq 0$.

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- Continuous functions

<u>Def</u> Let X and Y be metric spaces and let $f: X \to Y$ be a function.
We say f is continuous at $p \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all $x \in E$ with $d_X(x, p) < \delta$.
(In other words, if p is an interior point of $f^{-1}(B_\varepsilon(f(p)))$ for all $\varepsilon > 0$.)

We say f is continuous if it is continuous at every point.

<u>Thm</u> f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

proof) Clear from the definitions. ∎

<u>Thm</u> $f: X \to Y$ is continuous if and only if
for every open set $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open.

proof) ($\Rightarrow$) Suppose $f: X \to Y$ is continuous, and $V \subseteq Y$ is open.
If $p \in f^{-1}(V)$, then $f(p) \in V$, so $B_\varepsilon(f(p)) \subseteq V$ for some $\varepsilon > 0$.
Since f is continuous at p, p is an interior point of $f^{-1}(B_\varepsilon(f(p))) \subseteq f^{-1}(V)$.
Therefore $f^{-1}(V)$ is open.
($\Leftarrow$) Suppose $f^{-1}(V) \subseteq X$ is open, for every open set $V \subseteq Y$. Let $p \in X$.
Then, for every $\varepsilon > 0$, $f^{-1}(B_\varepsilon(f(p)))$ is open, so $p \in f^{-1}(B_\varepsilon(f(p)))$ is an interior point.
Hence there is $\delta > 0$ such that $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p)))$,

i.e. $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$.

Therefore f is continuous. ∎

<u>Thm</u> Let $f: X \to Y$ and $g: Y \to Z$ be maps between metric spaces.

Suppose f is continuous at $p \in X$ and g is continuous at $f(p)$.

Then $g \circ f$ is continuous at p.

proof) Let $\varepsilon > 0$. Since g is continuous at $f(p)$,

$$g\left(B_\eta(f(p))\right) \subseteq B_\varepsilon(g \circ f(p)) \quad \text{for some } \eta > 0.$$

Since f is continuous at p,

$$f(B_\delta(p)) \subseteq B_\eta(f(p)) \quad \text{for some } \delta > 0.$$

Hence

$$g \circ f(B_\delta(p)) \subseteq g(B_\eta(f(p))) \subseteq B_\varepsilon(g \circ f(p)),$$

and therefore $g \circ f$ is continuous at p. ∎

<u>Thm</u> Let f and g be complex continuous functions on a metric space X.

Then $f+g$, fg, $\frac{f}{g}$ are continuous on X.

assuming that $g(x) \neq 0$

proof) This easily follows from the last Corollary in p. 2 of this note. ∎

<u>Ex</u> Let $f: \mathbb{C} \to \mathbb{C}$ be given by a polynomial $f(z) = \sum_{n=0}^{d} c_n z^n$ for some $c_n \in \mathbb{C}$, $d \in \mathbb{N}$.

Then f is continuous.

<u>Ex</u> $\exp: \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$ is continuous.